Exercise 10

Evaluate the line integral, where C is the given curve.

$$\int_C y^2 z\, ds, \quad C \text{ is the line segment from } (3, 1, 2) \text{ to } (1, 2, 5)$$

Solution

The equation of the line going from $(3, 1, 2)$ to $(1, 2, 5)$ is

$$\begin{aligned}
\mathbf{y} &= \mathbf{m}t + \mathbf{b} \\
&= \langle 1-3, 2-1, 5-2\rangle t + \langle 3, 1, 2\rangle \\
&= \langle -2t, t, 3t\rangle + \langle 3, 1, 2\rangle \\
&= \langle 3-2t, 1+t, 2+3t\rangle,
\end{aligned}$$

where $0 \le t \le 1$. With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C y^2 z\, ds &= \int_0^1 [y(t)]^2 z(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^1 (1+t)^2 (2+3t) \sqrt{(-2)^2 + (1)^2 + (3)^2}\, dt \\
&= \sqrt{14} \int_0^1 (1+t)^2 (2+3t)\, dt \\
&= \sqrt{14} \int_0^1 (2 + 7t + 8t^2 + 3t^3)\, dt \\
&= \sqrt{14} \left(2t + \frac{7}{2}t^2 + \frac{8}{3}t^3 + \frac{3}{4}t^4\right)\Bigg|_0^1 \\
&= \sqrt{14} \left(2 + \frac{7}{2} + \frac{8}{3} + \frac{3}{4}\right) \\
&= \frac{107}{12}\sqrt{14}.
\end{aligned}$$